Lecture #2

Discrete Lotka-Volterra Model

Reference:


1 Two Model Species

Alfred James Lotka was a US mathematician, physical chemist, and statistician famous for his work in population dynamics and energetics. Vito Volterra was an Italian mathematician and physicist, best known for his contributions to mathematical biology. After World War I, Volterra turned his attention to the application of his mathematical ideas to biology, principally reiterating and developing the work of Pierre François Verhulst. The most famous outcome of this period is the Volterra-Lotka equations. One of the first models to incorporate interactions between predators and prey was proposed in 1925 by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra. Unlike the Malthusian and Logistic models we have previously seen, the Lotka-Volterra model is based on differential equations.

The Lotka-Volterra model describes interactions between two species in an ecosystem, a predator and a prey. This represents our first multi-species model. Since we are considering two species, the model will involve two equations, one which describes how the prey population changes and the second which describes how the predator population changes. For concreteness let us assume that the prey in our model are rabbits, and that the predators are foxes. If we let \( x_n \) and \( y_n \) represent the number of rabbits and foxes, respectively, that are alive at a discrete time \( n \), then the Lotka-Volterra model is:

\[
\begin{align*}
x_0, y_0 & \quad \text{are a given initial values} \\
x_{n+1} &= ax_n - bx_n y_n \\
y_{n+1} &= cy_n + dx_n y_n
\end{align*}
\]

where
- \( a \) is the natural growth rate of rabbits in the absence of predation,
- \( b \) is the death rate per encounter of rabbits due to predation,
- \( c \) is the efficiency of turning predated rabbits into foxes,
\( d \) is the natural death rate of foxes in the absence of food (rabbits).

The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics. Unfortunately, in its original form Lotka-Volterra has some significant problems. Neither equilibrium point is stable. Instead the predator and prey populations seem to cycle endlessly without settling down quickly. It can be shown that this behavior will be observed for any set of values of the model’s four parameters. While this cycling has been observed in nature, it is not overwhelmingly common. It appears that Lotka-Volterra by itself is not sufficient to model many predator-prey systems.

For example, for the initial population numbers \( x_0 = 100 \) and \( y_0 = 10 \) and
\[
a = 1.073; \quad b = 0.006; \quad c = 0.9; \quad d = 0.0021
\]
we get:

\[
\begin{array}{ccc}
\hline
n & x_n & y_n \\
\hline
0 & 100 & 10 \\
1 & 101.3000000 & 11.10000000 \\
2 & 101.9483200 & 12.35130300 \\
3 & 101.8353798 & 13.76048134 \\
4 & 100.8615395 & 15.32717128 \\
5 & 98.94889932 & 17.04089054 \\
6 & 96.05510479 & 18.87777395 \\
7 & 92.18728811 & 20.79794032 \\
8 & 87.41312585 & 22.74448829 \\
9 & 81.86528313 & 24.64518978 \\
10 & 75.73593616 & 26.41760022 \\
11 & 69.26008940 & 27.97743974 \\
12 & 62.68975606 & 29.24890772 \\
13 & 56.26446692 & 30.17459141 \\
\hline
\end{array}
\]

For example, for the initial population numbers \( x_0 = 100 \) and \( y_0 = 0 \) and the same parameter as above:
\[
\lim_{n \to \infty} x_n = +\infty
\]

For example, for the initial population numbers \( x_0 = 0 \) and \( y_0 = 10 \) and the same parameter as above:
\[
\lim_{n \to \infty} y_n = 0
\]

2 Questions

2.1

Explain the following critical cases:
• $x_0 = 100$ and $y_0 = 0$.
• $x_0 = 0$ and $y_0 = 10$.
• $a = 1$.
• $b = 0$.
• $c = 1$.
• $d = 0$.

2.2

for the initial population numbers $x_0 = 150$ and $y_0 = 20$ and

$$a = 1.06; \quad b = 0.007; \quad c = 0.85; \quad d = 0.003$$

Compute $x_n$ and $y_n$ for $n = 1, 2, 3, 4, 5$. 